

Module III

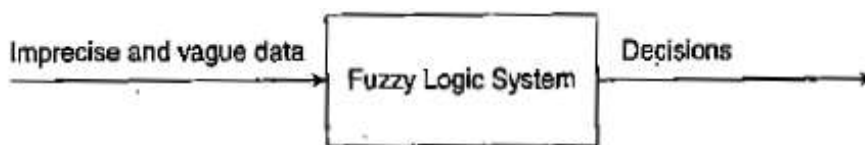
Fuzzy logic - fuzzy sets - properties - operations on fuzzy sets, fuzzy relations - operations on fuzzy relations.

FUZZY LOGIC

Fuzzy logic is a form of multi-valued logic to deal with reasoning that is approximate rather than precise. Fuzzy logic variables may have a truth value that ranges between 0 and 1.

Fuzzy logic offers soft computing paradigm the important concept of computing with words. It provides a technique to deal with imprecision and information granularity.

The fuzzy theory provides a mechanism for representing linguistic constructs such as "high," "low," "medium," "tall," "many." In general, fuzzy logic provides an inference structure that enables appropriate human reasoning capabilities. The theory of Fuzzy logic is based upon the notion of relative graded membership and so are the functions of cognitive processes. The utility of fuzzy sets lies in their ability to model uncertain or ambiguous data and to provide suitable decisions as in Figure:



In *fuzzy* systems, values are indicated by a number (called a truth value) ranging from 0 to 1, where 0.0 represents absolute falseness and 1.0 represents absolute truth.

Fuzzy sets that represent fuzzy logic provide means to model the uncertainty associated with vagueness, imprecision and lack of information regarding a problem or a plant or a system, etc.

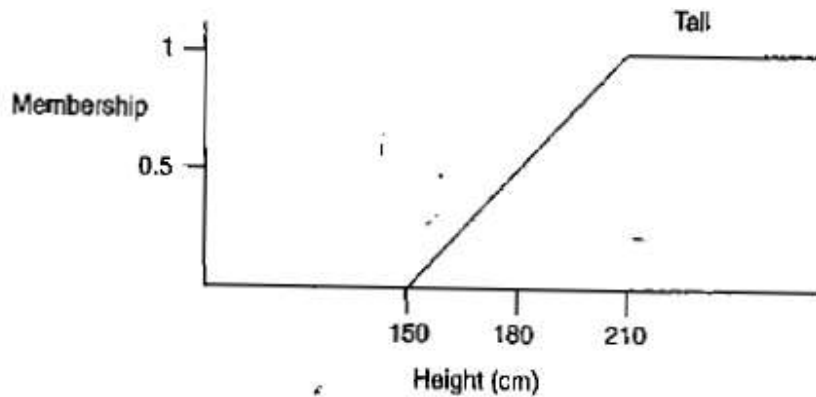
Consider the meaning of a "short person". For an individual X , a short person may be one whose height is below 4' 25". For other individual Y , a short person may be one whose height is below or equal to 3'90". The word "short" is called a linguistic descriptor. The term "short" provides the same meaning to individuals X and Y , but it can be seen that they both do not provide a unique definition. This variable "short" is called as *linguistic variable* which represents the imprecision existing in the system.

The basis of the theory lies in making the membership function lie over a range of real numbers from 0.0 to 1.0. The fuzzy set is characterized by (0.0, 0, and 1.0). The membership value is "1" if it belongs to the set and "0" if it is not a member of the set. Thus membership in a set is found to be binary, that is, either the element is a member of a set or not. It can be indicated as

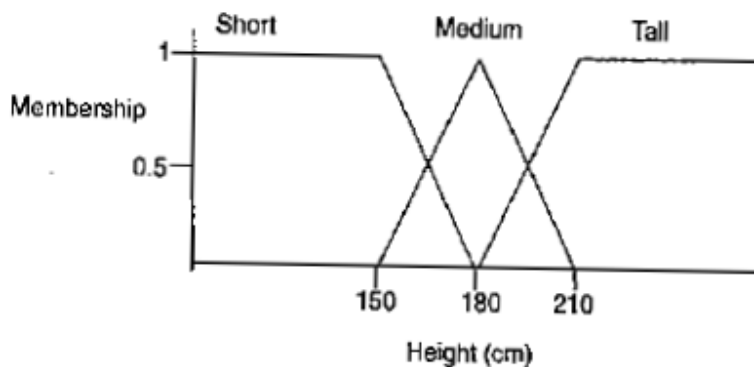
$$\chi_A(x) = \begin{cases} 1, & x \in A \\ 0, & x \notin A \end{cases}$$

$\chi_A(x)$ is the membership of element x in the set A and A is the entire set on the universe.

Fuzzy logic operates on the concept of membership. In Figure below, the objective term "tall" has been assigned fuzzy values. At 150 cm and below, a person does not belong to the fuzzy class while for above 180, the person certainly belong to category "tall." However, between 150 and 180 cm, the degree of membership for the class "tall" can be assigned from the curve linearly between 0 and 1.



The fuzzy concept "tallness" can be extended into "short," "medium" and "tall" as shown in figure below:



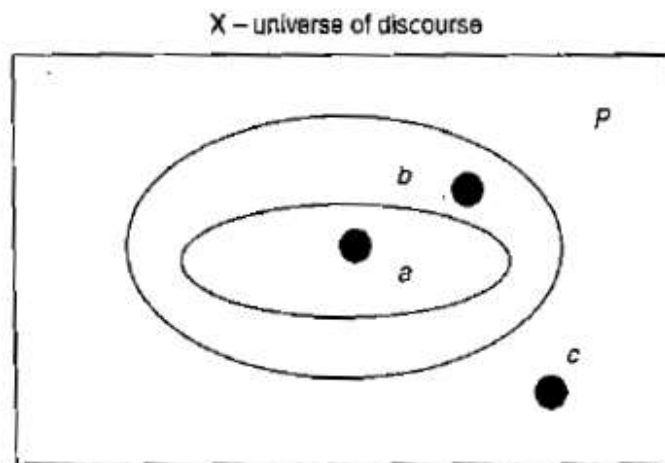
The membership was extended to possess various "degrees of membership" on the real continuous interval [0, 1]. The degree of membership of any particular element of a fuzzy set expresses the degree of compatibility of the element with a concept represented by fuzzy set.

It means that a fuzzy set A contains an object x to degree $a(x)$, that is, $a(x) = \text{Degree}(x \in A)$, and the map $a: X \rightarrow \{\text{Membership degree}\}$ is called a *set function* or a *membership function*. The fuzzy set A can be expressed as $A = \{(x, a(x))\}$.

Fuzziness describes the ambiguity of an event and randomness describes the uncertainty in the occurrence of an event.

Boundary Region of a fuzzy set

From Figure below it can be noted that " a " is clearly a member of fuzzy set P , " c " is clearly not a member of fuzzy set P and the membership of " b " is found to be vague. Hence " a " can take membership value 1, " c " can take membership value 0 and " b " can take membership value between 0 and 1 [0 to 1], say 0.4, 0.7, etc. This is said to be a partial membership of fuzzy set P .

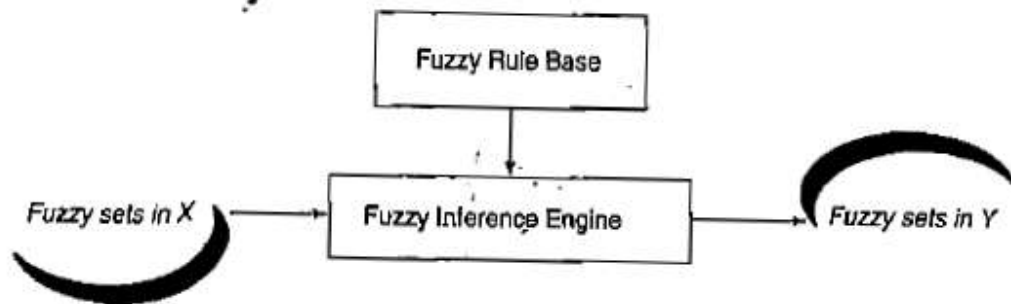


The membership function for a set maps each element of the set to a membership value between 0 and 1 and uniquely describes that set. The value 0 and 1 describes not belonging to and belonging to a set, respectively; values in between represents "fuzziness."

Fuzzy sets form the building blocks for fuzzy *IF-THEN* rules which have the general form "*IF X is A THEN Y is B*," where A and B are fuzzy sets. The term "fuzzy systems" refers mostly to systems that are governed by fuzzy IF-THEN rules.

The IF part of an implication is called the *antecedent* whereas the THEN part is called a *consequent*. A fuzzy system is a set of fuzzy rules that converts

inputs to outputs. The basic configuration of a pure fuzzy system is shown in figure below.



FUZZY SETS

Fuzzy sets may be viewed as an extension and generalization of the basic concepts of crisp sets.

Fuzzy set it allows partial membership. A fuzzy set is a set having degrees of membership between 1 and 0. The membership in a fuzzy set need not be complete, i.e., member of one fuzzy set also be member of other fuzzy sets in the same universe.

Vagueness is introduced in fuzzy set by eliminating the sharp boundaries that divide members from nonmembers in the group. There is a gradual transition between full membership and non membership, not abrupt transition.

A fuzzy set *in* the universe of discourse U can be defined as a set of ordered pairs and it is given by,

$$A = \{(x, \mu_A(x)) \mid x \in U\}$$

$\mu_A(x)$ is the degree of membership of x in A and it indicates the degree that x belongs to A .

The degree of membership $\mu_A(x)$ assumes values in the range from 0 to 1, i.e., the membership is set to unit interval $[0, 1]$ or $\mu_A(x) \in [0, 1]$.

The universe of discourse U is discrete and finite fuzzy set A is given as follows:

$$A = \left\{ \frac{\mu_A(x_1)}{x_1} + \frac{\mu_A(x_2)}{x_2} + \frac{\mu_A(x_3)}{x_3} + \dots \right\} = \left\{ \sum_{i=1}^n \frac{\mu_A(x_i)}{x_i} \right\}$$

A fuzzy set is universal fuzzy set if and only if the value of the membership function is 1 for all the members under consideration. Any fuzzy set A defined on a universe U is a subset of that universe.

Two fuzzy sets A and B are said to be equal fuzzy sets if $\mu_A(x) = \mu_B(x)$ for all $x \in U$.

A fuzzy set A is said to be empty fuzzy set if and only if the value of the membership function is 0 for all possible members considered. The universal fuzzy set can also be called whole fuzzy set.

The collection of all fuzzy sets and fuzzy subsets on universe U is called, fuzzy power set $P(U)$. Since all the fuzzy sets can overlap, the cardinality of the fuzzy power set, $n_{P(U)}$ is infinite. i.e.) $n_{P(U)} = \infty$.

PROPERTIES OF FUZZY SETS

1. Commutativity

$$A \cup B = B \cup A; \quad A \cap B = B \cap A$$

2. Associativity

$$A \cup (B \cup C) = (A \cup B) \cup C$$

$$A \cap (B \cap C) = (A \cap B) \cap C$$

3. Distributivity

$$A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$$

$$A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$$

4. Idempotency

$$A \cup A = A; \quad A \cap A = A$$

5. Identity

$$A \cup \phi = A \quad \text{and} \quad A \cup U = U(\text{universal set})$$
$$A \cap \phi = \phi \quad \text{and} \quad A \cap U = A$$

6. Involution (double negation)

$$\overline{\overline{A}} = A$$

7. Transitivity

$$\text{If } A \subseteq B \subseteq C, \text{ then } A \subseteq C$$

8. De Morgan's law

$$\overline{A \cup B} = \overline{A} \cap \overline{B}; \quad \overline{A \cap B} = \overline{A} \cup \overline{B}$$

FUZZY SET OPERATIONS

Let A and B be fuzzy sets in the universe of discourse U . For a given element x on the universe, the following function theoretic operations of union, intersection and complement are defined for fuzzy sets A and B on U .

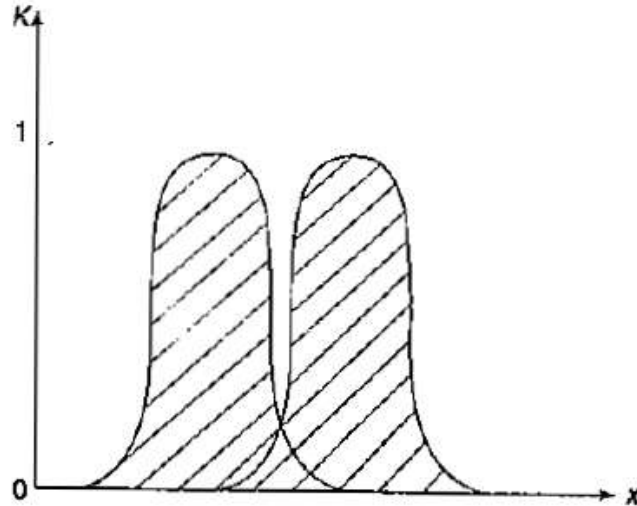
1) Union

The union of fuzzy sets A and B , denoted by $A \cup B$, is defined as

$$\mu_{A \cup B}(x) = \max[\mu_A(x), \mu_B(x)] = \mu_A(x) \vee \mu_B(x) \quad \text{for all } x \in U$$

Where \vee indicates max operation.

The Venn diagram for union operation of fuzzy sets \underline{A} and \underline{B} is shown in Figure below:

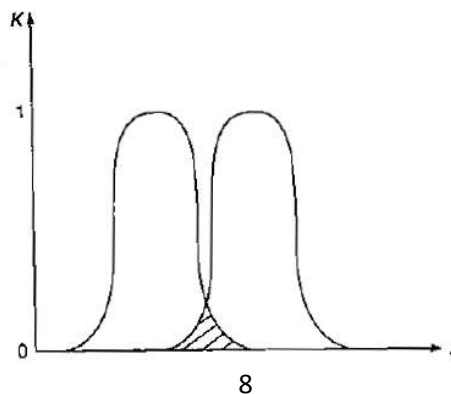


2) Intersection

The intersection of fuzzy sets \underline{A} and \underline{B} , denoted by $\underline{A} \cap \underline{B}$, is defined by

$$\mu_{A \cap B}(x) = \min[\mu_A(x), \mu_B(x)] = \mu_A(x) \wedge \mu_B(x) \quad \text{for all } x \in U$$

where \wedge indicates min operator. The Venn diagram for intersection operation of fuzzy sets \underline{A} and \underline{B} is shown in Figure below.

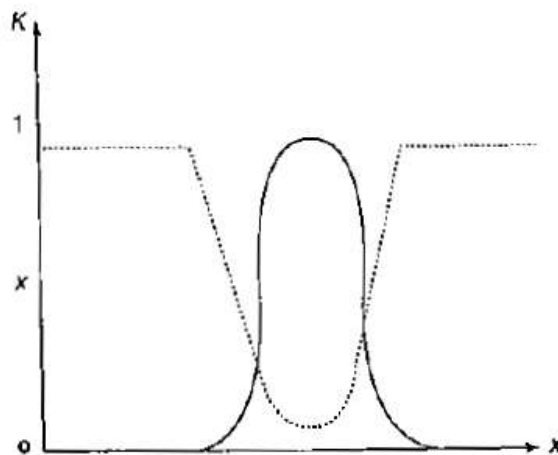


3) Complement

When $\mu_A(x) \in [0,1]$, the complement of A , denoted as \bar{A} is defined by

$$\mu_{\bar{A}}(x) = 1 - \mu_A(x) \text{ for all } x \in U$$

The Venn diagram for complement operation of fuzzy set A is shown in Figure below.



4) Algebraic sum

The algebraic sum ($A + B$) of fuzzy sets, fuzzy set A and B is defined as

$$\mu_{A+B}(x) = \mu_A(x) + \mu_B(x) - \mu_A(x) \cdot \mu_B(x)$$

5) Algebraic product

The algebraic product ($A \cdot B$) of two fuzzy sets A and B is defined as

$$\mu_{A \cdot B}(x) = \mu_A(x) \cdot \mu_B(x)$$

6) Bounded sum

The bounded sum ($A \oplus B$) of two fuzzy sets A and B is defined as

$$\mu_{A \oplus B}(x) = \min\{1, \mu_A(x) + \mu_B(x)\}$$

7. Bounded difference

The bounded difference ($A \ominus B$) of two fuzzy sets A and B is defined as

$$\mu_{A \ominus B}(x) = \max\{0, \mu_A(x) - \mu_B(x)\}$$

FUZZY RELATIONS

Fuzzy relations relate elements of one universe (say X) to those of another universe (say Y) through the Cartesian product of the two universes. These can also be referred to as fuzzy sets defined on universal sets, which are Cartesian products.

A fuzzy relation is based on the concept that everything is related to some extent or unrelated.

A fuzzy relation is a fuzzy set defined on the Cartesian product of classical sets $\{X_1, X_2, \dots, X_n\}$ where tuples (x_1, x_2, \dots, x_n) may have varying degrees of membership $\mu_R(x_1, x_2, \dots, x_n)$ within the relation. That is,

$$R(X_1, X_2, \dots, X_n) = \int_{X_1 \times X_2 \times \dots \times X_n} \mu_R(x_1, x_2, \dots, x_n) | (x_1, x_2, \dots, x_n), x_i \in X_i$$

A fuzzy relation between two sets X and Y is called binary fuzzy relation and is denoted by $R(X, Y)$. A binary relation $R(X, Y)$ is referred to as bipartite graph when $X \neq Y$. The binary relation on a single set X is called directed graph or digraph. This relation occurs when $X=Y$ and is denoted as $R(X, X)$ or $R(X^2)$.

Fuzzy Matrix

Let

$$X = \{x_1, x_2, \dots, x_n\} \quad \text{and} \quad Y = \{y_1, y_2, \dots, y_m\}$$

Fuzzy relation $R(X, Y)$ can be expressed by an $n \times m$ matrix as follows:

$$R(X, Y) = \begin{bmatrix} \mu_R(x_1, y_1) & \mu_R(x_1, y_2) & \dots & \mu_R(x_1, y_m) \\ \mu_R(x_2, y_1) & \mu_R(x_2, y_2) & \dots & \mu_R(x_2, y_m) \\ \dots & \dots & \dots & \dots \\ \mu_R(x_n, y_1) & \mu_R(x_n, y_2) & \dots & \mu_R(x_n, y_m) \end{bmatrix}$$

The matrix representing a fuzzy relation is called fuzzy matrix. A fuzzy relation R is a mapping from Cartesian space $X \times Y$ to the interval $[0, 1]$.

Fuzzy Graph

A fuzzy graph is a graphical representation of a binary fuzzy relation. Each element in X and Y corresponds to a node in the fuzzy graph. The connection links are established between the nodes by the elements of $X \times Y$ with nonzero membership grades in $R(X, Y)$. The links may also be present in the form of arcs.

When $X \neq Y$, the link connecting the two nodes is an undirected binary graph called bipartite graph.

When $X = Y$, a node is connected to itself and directed links are used; in such a case, the fuzzy graph is called directed graph.

The domain of a binary fuzzy relation $R(X, Y)$ is the fuzzy set, $dom R(X, Y)$, having the membership function as

$$\mu_{\text{domain } R}(x) = \max_{y \in Y} \mu_R(x, y) \quad \forall x \in X$$

Consider a universe $X = \{x_1, x_2, x_3, x_4\}$ and the binary fuzzy relation on X as

$$R(X, X) = \begin{matrix} & \begin{matrix} x_1 & x_2 & x_3 & x_4 \end{matrix} \\ \begin{matrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{matrix} & \begin{bmatrix} 0.2 & 0 & 0.5 & 0 \\ 0 & 0.3 & 0.7 & 0.8 \\ 0.1 & 0 & 0.4 & 0 \\ 0 & 0.6 & 0 & 1 \end{bmatrix} \end{matrix}$$

The bipartite graph and simple fuzzy graph of $R(X, X)$ is shown in Figures below:

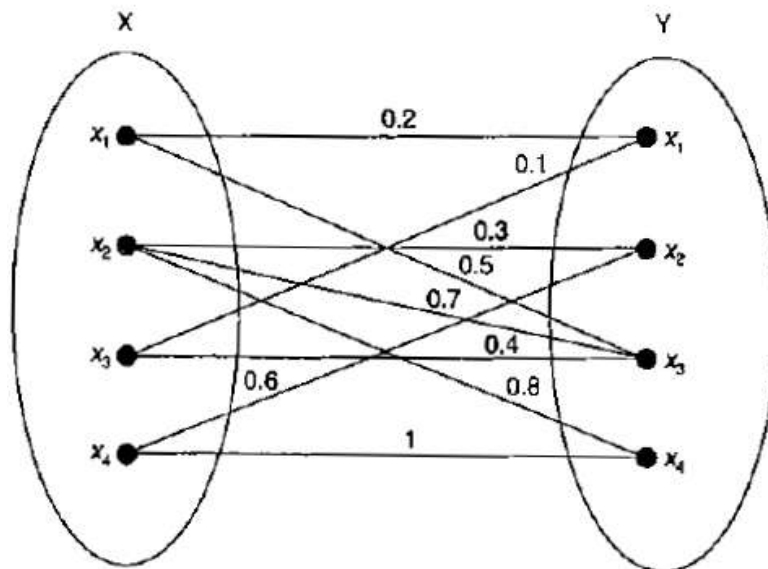


Fig: Bipartite graph

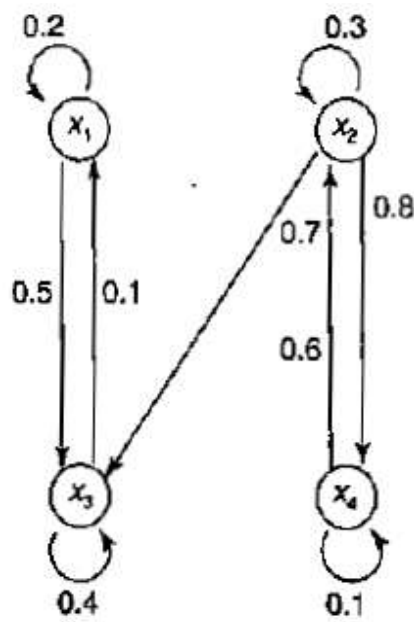


Fig: Simple fuzzy graph.

Let

$$\underline{X} = \{x_1, x_2, x_3, x_4\} \text{ and } \underline{Y} = \{y_1, y_2, y_3, y_4\}$$

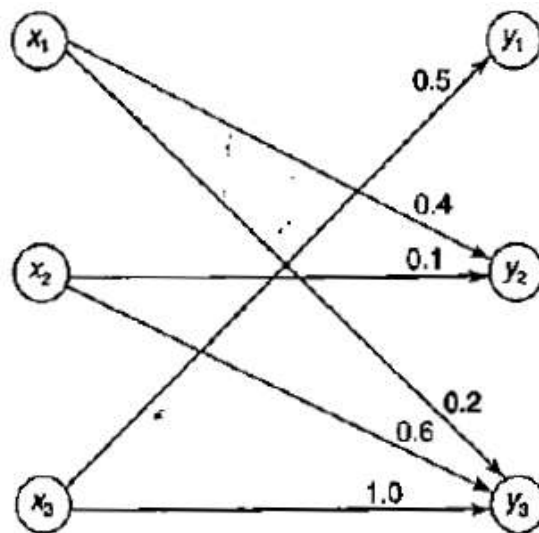
Let \mathcal{R} be a relation from \underline{X} to \underline{Y} given by

$$\mathcal{R} = \frac{0.2}{(x_1, y_3)} + \frac{0.4}{(x_1, y_2)} + \frac{0.1}{(x_2, y_2)} + \frac{0.6}{(x_2, y_3)} + \frac{1.0}{(x_3, y_3)} + \frac{0.5}{(x_3, y_1)}$$

The corresponding fuzzy matrix for relation \mathcal{R} is

$$\mathcal{R} = \begin{matrix} & \begin{matrix} y_1 & y_2 & y_3 \end{matrix} \\ \begin{matrix} x_1 \\ x_2 \\ x_3 \end{matrix} & \begin{bmatrix} 0 & 0.4 & 0.2 \\ 0 & 0.1 & 0.6 \\ 0.5 & 0 & 1.0 \end{bmatrix} \end{matrix}$$

The graph of the above relation $\mathcal{R} = \underline{X} \times \underline{Y}$ is shown below:



OPERATIONS ON FUZZY RELATIONS

The basic operations on fuzzy sets also apply on fuzzy relations.

Let \mathcal{R} and \mathcal{S} be fuzzy relations on the Cartesian space $X \times Y$. The operations that can be performed on these fuzzy relations are described below:

1. Union

$$\mu_{\mathcal{R} \cup \mathcal{S}}(x, y) = \max[\mu_{\mathcal{R}}(x, y), \mu_{\mathcal{S}}(x, y)]$$

2. Intersection

$$\mu_{\mathcal{R} \cap \mathcal{S}}(x, y) = \min[\mu_{\mathcal{R}}(x, y), \mu_{\mathcal{S}}(x, y)]$$

3. Complement

$$\mu_{\bar{\mathcal{R}}}(x, y) = 1 - \mu_{\mathcal{R}}(x, y)$$

4. Containment

$$\mathcal{R} \subset \mathcal{S} \Rightarrow \mu_{\mathcal{R}}(x, y) \leq \mu_{\mathcal{S}}(x, y)$$

5. Inverse

The inverse of a fuzzy relation R on $X \times Y$ is denoted by R^{-1} . It is a relation on $Y \times X$ defined by,

$$R^{-1}(y, x) = R(x, y) \text{ for all pairs } (y, x) \in Y \times X.$$

6. Projection

For a fuzzy relation $R(X, Y)$, let $(R \downarrow Y)$ denote the projection of R onto Y . Then $(R \downarrow Y)$ is a fuzzy relation in Y whose membership function is defined by:

$$\mu_{(R \downarrow Y)}(x, y) = \max_x \mu_{\mathcal{R}}(x, y)$$